ON MAXIMUM THRUST NOZZLES WITH ARBITRARY ISOPERIMETRIC CONDITIONS

(O SOPLAKH MAKSIMAL'NOI TIAGI S PROIZVOL'NYMI IZOPERIMETRICHESKIMI USLOVIIAMI)

PMM Vol.28, № 1, 1964, pp.182-183

V.M. BORISOV and A.V. SHIPILIN
(Moscow)

(Received October 24, 1963)

Guderley and Armitage [1] obtained necessary conditions for an extremum in the problem of nozzles with the greatest thrust under arbitrary conditions imposed on the nozzle wall. Numerical solution of this problem is associated with a very complicated boundary value problem for nonlinear partial differential equations. In the present paper we find one class of solutions of this boundary value problem.

1. In [1] the following problem was considered. We are given (see Fig.1) the characteristic of the free stream ab and the external pressure p_0. It is required to find the nozzle contour ac possessing the maximum thrust with certain restrictions. For example, such restrictions may consist of a given area of the side surfaces of the nozzle, a given volume of the working section of the nozzle, and so on. Let the characteristic of the first family, passing through the point c, be represented by the line bc. The problem is formulated in the following manner.

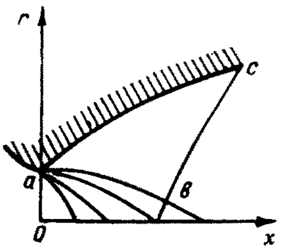

Fig. 1

For a given pressure p_0 and a given starting characteristic ab, to find the function $\eta(x)$ realizing the extremum of the functional

$$\chi = \int_{x_a}^{x_c} \{p\,[x, \eta\,(x)] - p_0\}\, \eta\eta'\, dx$$

with the isoperimetric conditions on ac

$$s^i = \int_{x_a}^{x_c} f^i\,\{u[x, \eta\,(x)];\ v[x,\eta(x)];\ \eta(x);\ \eta'(x);\ x;\ p_0\}\, dx \tag{1}$$

the differential relations on ac

$$\eta'\,(x)\, u - v = 0$$

the differential relations in the triangle abc

$$\frac{\partial u}{\partial r} - \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial r\rho u}{\partial x} + \frac{\partial r\rho v}{\partial r} = 0 \tag{2}$$

and the relations

$$\frac{\partial p}{\partial u} = -\rho u, \qquad \frac{\partial p}{\partial v} = -\rho v, \qquad a^2 = \varkappa \frac{p}{\rho}, \qquad a^2 = \frac{dp}{d\rho}, \qquad \sin^2 \alpha = \frac{a^2}{u^2 + v^2}$$

Here x and r are Cartesian coordinates in the meridional plane of the axisymmetric flow, u and v are the projections of the velocity on the axes of x and r, p is the pressure, ρ is the density, a is the velocity of sound, α is the Mach angle, and $\varkappa$ is the constant adiabatic exponent.

Necessary conditions for an extremum of χ were obtained in [1] and have the form

in the triangle abc

$$\frac{\partial h_1}{\partial r} + r\rho\left(1 - \frac{u^2}{a^2}\right)\frac{\partial h_2}{\partial x} - \frac{r\rho uv}{a^2}\frac{\partial h_2}{\partial r} = 0, \qquad \frac{\partial h_1}{\partial x} + \frac{r\rho uv}{a^2}\frac{\partial h_2}{\partial x} - r\rho\left(1 - \frac{v^2}{a^2}\right)\frac{\partial h_2}{\partial r} = 0 \tag{3}$$

on the nozzle contour ac

$$h_1 = \eta\rho v - c_1^{\,i}\frac{u}{u^2 + v^2}(f_u^{\,i} u + f_v^{\,i} v) \tag{4}$$

$$h_2 = u - \frac{c_1^{\,i}}{\eta\rho}\left\{f_v^{\,i} - \frac{v}{u^2 + v^2}(f_u^{\,i} u + f_v^{\,i} v) + \right. \tag{5}$$

$$\left. + \eta\rho \int_x^{x_c} \frac{1}{\eta\rho u}\left[f_\eta^{\,i} + f_u^{\,i}\frac{\partial u}{\partial r} + f_v^{\,i}\frac{\partial v}{\partial r} - \frac{d}{dx} f_{\eta'}^{\,i}\right]dx\right\} + c_3$$

on the concluding characteristic bc

$$h_1 + h_2 r\rho \cot \alpha = 0 \tag{6}$$

on the point c

$$(p - p_0)\,\eta\eta' + c_1^{\,i} f^i = 0 \tag{7}$$

$$(p - p_0)\,\eta + c_1^{\,i} f_{\eta'}^{\,i} + \eta\rho u\left\{c_3 + u - \frac{c_1^{\,i}}{\eta\rho}\left[f_v^{\,i} - \frac{v}{u^2 + v^2}(f_u^{\,i} u + f_v^{\,i} v)\right]\right\} = 0 \tag{8}$$

Here $h_1(x,r)$ and $h_2(x,r)$ are Lagrange multipliers, corresponding to the differential relations (2), $c_1^{\,i}$ are constant Lagrange multipliers related to the isoperimetric conditions (1), and c_3 is a constant of integration.

For supersonic flows system (3) is of hyperbolic type and its characteristic directions coincide with the characteristic directions of the system (2). The condition of compatibility for h_1 and h_2 on the characteristic bc has the form

$$dh_1 - dh_2\, r\rho \cot \alpha = 0 \tag{9}$$

2. Let us discard some of the restrictions (1) of the original variational problem. It is obvious that the solution of the simplified problem is simultaneously a solution of the original problem if the value of the numbers s^i for the discarded restrictions are specified according to the solution of the simplified problem. This enables one to construct simple examples of solutions of the original problem.

Suppose that of all the restrictions we retain, for example, only the restriction of the value of p_0. It is known that the nozzle with uniform flow at the exit gives maximum thrust for a given value of p_0. Let us calculate, for example, the area of the side surface of the nozzle thus found. The nozzle obtained has maximum thrust of all nozzles with the same side surface area and external pressure p_0. Similar arguments can be employed also for the restrictions on the values of p_0 and x_c, or for x_c and r_c.

For all these examples necessary conditions for the extremum of the original problem must pass over into the known conditions for extremalcy on the concluding characteristic bc with $c_1^{\,i} = 0$. In fact, it is not difficult to verify that Equations

$$h_1 = \eta \rho v, \qquad h_2 = u + c_2 \tag{10}$$

are an integral of the system (3). From conditions (6), (9) and (10) we find that along the concluding characteristic bc the relations

$$r\rho v^2 \tan\alpha = \text{const}, \qquad u + v\tan\alpha = -c_2$$

are fulfilled.

These relations were obtained in [2] from consideration of the variational problem for fixed points a and c.

Condition (8) with $c_1' = 0$ passes over into the well known Busemann condition.

BIBLIOGRAPHY

1. Guderley, K.G. and Armitage, J.V., A General Method for the Determination of Best Supersonic Rocket Nozzles. Paper Presented at the Symposium on Extremal Problems in Aerodynamics, Boeing Scientific Research Laboratories, Flight Sciences Laboratory, Seattle, Washington, December 3-4, 1962.
2. Shmyglevskii, Iu.D., Nekotorye variatsionnye zadachi gazovoi dinamiki osesimmetrichnykh sverkhzvukovykh techenii (Certain variational problems of gas dynamics of axisymmetric flows). *PMM* Vol.21, № 2, 1957.

Translated by A.H.A.